INVESTIGATION OF THE FREE BOUNDARY NEIGHBORHOOD OF A GAS IN MOTION BEHIND A DETONATION WAVE PROPAGATING IN A SPACE WITH A CONICAL CUTOUT

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A space containing an explosive substance is provided with a conical cutout having its vertex at point O. As the result of ignition at point O a detonation wave and the motion of products of explosion behind it are generated. Presence of the hollow cone originates a rarefaction wave bounded by a free surface. The behavior of gas-dynamic functions in this rarefaction wave is analyzed in this paper in terms of the cone vertex angle and of the adiabatic exponent of products of explosion.

The products of the explosion initiated by ignition at point O of the space containing an explosive substance, and provided with a cutout in the form of a hollow cone S with its vertex at point O are bounded by the detonation wave front and the free surface. The detonation wave front propagates with constant velocity D. At an instant t the detonation wave front is represented by that part of sphere Q of radius Dt having its center at O, which is bounded by its intersection with cone S along circle M_t. In the following the form of the free surface which depends on the adiabatic exponent $\varkappa$ of the products of explosion and on the cone S vertex angle γ will be determined.

Critical values of angle $\gamma_k(\varkappa)$ will be determined for each value of the adiabatic exponent $\varkappa$. If exponent $\varkappa \geq 2$ and angle $\gamma \geq \gamma_k(\varkappa)$ (conversely $\varkappa \leq 2$ and $\gamma \leq \gamma_k(\varkappa)$) then at the instant t the free surface is defined by a cone the base of which coincides with circle M_t, and its vertex lies on the axis of the cone S. In all other cases the free surface consists of a truncated cone adjoining the detonation wave front, and of a surface of revolution generated by a line of nonzero curvature. The latter indicates the onset of a stream with a base expanding with time.

1. Statement of problem. The motion of gas behind a detonation wave propagating from the ignition initiation point O (the coordinate origin) in a space provided with a conical cutout (the cone axis is defined by $x = 0$, $y = 0$, $z \leq 0$, and γ is the angle between the cone axis and its generatrix) and filled with an explosive substance, is cylindrically symmetric and self-similar. In view of this cylindrical symmetry it is sufficient to consider the flow in the rz-plane ($r = \sqrt{x^2 + y^2}$). The number of independent variables in the equations of gas dynamics which define the flow by virtue of self-similarity) may be reduced to two, viz. $\xi = r/t$, $\eta = z/t$. The unknown functions are: the density ρ and two velocity components V_r and V_z along the axes r and z. The pressure p of the products of explosion is defined by the equation of state

$$p = \varkappa\rho^{\varkappa} \tag{1.1}$$

where $\varkappa$ is the adiabatic exponent. At the instant immediately preceding ignition

$$t=0, \quad p=0, \quad v_r=v_z=0, \quad \rho=\varkappa/(\varkappa+1) \tag{1.2}$$

The detonation wave satisfies the Jouguet condition, and at instant t is represented by the part of a sphere of radius DT ($D = \varkappa + 1$) bounded by the corresponding cone section. The arc of circle AA' of radius D with its center at O and symmetric relative to the η-axis corresponds in the $\xi\eta$-plane to the detonation wave front. The angle between radius OA (or OA') and the semi-axis η ($\eta \leq 0$) is obviously equal to γ. For the purpose of this analysis it is convenient to introduce polar coordinates α and δ with the origin at point A

$$\xi = D\sin\gamma + \alpha\cos(\gamma + \sqrt{h}\,\delta),$$

$$\eta = -D\cos\gamma + \alpha\sin(\gamma + \sqrt{h}\,\delta) \quad (h = (\varkappa+1)/(\varkappa-1)) \tag{1.3}$$

and velocity components v_α and v_δ

$$v_r = D\sin\gamma + v_\alpha\cos(\gamma+\sqrt{h}\,\delta) - v_\delta\sin(\gamma+\sqrt{h}\,\delta)$$

$$v_z = -D\cos\gamma + v_\alpha\sin(\gamma+\sqrt{h}\,\delta) + v_\delta\cos(\gamma+\sqrt{h}\,\delta)$$

where $\sqrt{h}\,\delta$ is the angle between the half-line emanating from point A and the direction of the detonation wave front at that point, α is the distance from point A along the half-line, and v_α and v_δ are the velocity components along the half-line emanating from point A and a line parpendicular to the latter respectively.

The Eqs. of gas dynamics in terms of these variables are of the form

$$(v_\alpha-\alpha)\frac{\partial v_\alpha}{\partial\alpha} + \frac{v_\delta}{\alpha}\left(\frac{1}{\sqrt{h}}\frac{\partial v_\alpha}{\partial\delta} - v_\delta\right) + \frac{1}{\varkappa-1}\frac{\partial f}{\partial\alpha} = 0$$

$$(v_\alpha-\alpha)\frac{\partial v_\delta}{\partial\alpha} + \frac{v_\delta}{\alpha}\left(\frac{1}{\sqrt{h}}\frac{\partial v_\delta}{\partial\delta} + v_\alpha\right) + \frac{1}{(\varkappa-1)\sqrt{h}\,\alpha}\frac{\partial f}{\partial\delta} = 0$$

$$(v_\alpha-\alpha)\frac{\partial f}{\partial\alpha} + \frac{v_\delta}{\sqrt{h}\,\alpha}\frac{\partial f}{\partial v_\delta} + (\varkappa-1)f\left[\frac{\partial v_\alpha}{\partial\alpha} + \frac{v_\alpha}{\alpha} + \frac{1}{\sqrt{h}\,\alpha}\frac{\partial v_\delta}{\partial\alpha} + \right.$$
$$\left. + \frac{v_\alpha\cos(\gamma+\sqrt{h}\,\delta) - v_\delta\sin(\gamma+\sqrt{h}\,\delta) + D\sin\gamma}{\alpha\cos(\gamma+\sqrt{h}\,\delta) + D\sin\gamma}\right] = 0 \tag{1.5}$$

Here f is the square of the velocity of sound.

A domain G in the $\xi\eta$-plane in which the flow behind the wave coincided with the spherically symmetric solution derived by Zel'dovich [2] was determined in paper [1]. This domain is bounded by the detonation wave front AA' and two symmetric segments of characteristics AB and $A'B'$. For sufficiently large angles γ ($\gamma > \gamma_\varkappa$) points B and B' coincide, and lie on the η-axis. Otherwise ($\gamma < \gamma_\varkappa$), the shock wave front BB' which connects points B and B' (Fig. 1, *a* and *b*) also penetrated domain G. The magnitude of angle $\gamma_\varkappa$ depends on $\varkappa$ only, and is determined by numerical integration.

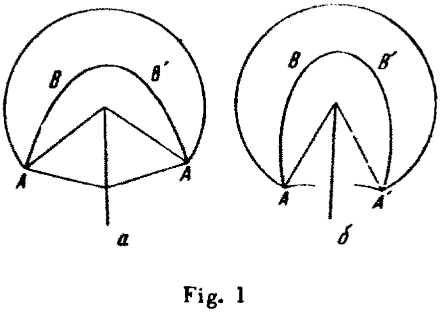

Fig. 1

In the following asymptotic behavior of gasdynamic functions is analyzed outside domain G, namely, in the vicinity of point A and of the free boundary emanating from that point (because of symmetry there is no need to repeat this analysis for point A'). The presence of a hollow cone creates at point A a centered rarefaction wave bounded by the free boundary, i.e. by a line along which $p = 0$. This wave corresponds as regards its predominant term to the Prandtl-Meyer solution

$$v_\alpha = \varkappa \sqrt{h} \sin\delta, \qquad v_\delta = \varkappa \cos\delta, \qquad f = \varkappa \cos^2 \delta \tag{1.6}$$

while the half-line emanating from point A at the angle $\frac{1}{2}\sqrt{h}\,\pi$ corresponds in the $\xi\eta$-plane to the detonation wave front(*). Characteristic AB which bounds domain G does not however belong to the centered rarefaction wave, because the values of parameters along characteristic AB coincide with those of (1.6) at the point A only. This shows that in the immediate vicinity of characteristic AB the solution structure is somewhat different when α is of the order of δ^3 (the equation of characteristic AB is $\alpha \approx {}^{64}/_{81}\, \varkappa h^{1.5}\, \delta^3$). Namely, there exists in the vicinity of the point A, between the characteristic AB and the rarefaction wave, a transition zone where the solution is of the form

$$v_\alpha = v_{\alpha_0}(\psi)\delta + v_{\alpha_1}(\psi)\delta^3 + \ldots,\ v_\delta = v_{\delta_0}(\psi) + v_{\delta_1}(\psi)\delta^2 + \ldots$$

$$f = f_0(\psi) + f_1(\psi)\delta^2 + \ldots,\ \psi = \alpha / \delta^3 \quad (\alpha \to 0,\ \delta \to 0) \tag{1.7}$$

Expression (1.7) corresponds to a split of singularity in the neighborhood of point $\alpha = 0$, $\delta = 0$ in the $\alpha\delta$-plane with the upper boundary $\psi = \psi_0 = {}^{64}/_{81}\, \varkappa h^{1.5}$ being the characteristic AB, and the lower boundary $\psi = 0$ indicating a transition to finite values of δ (Fig. 2). Asymptotics (1.7) were derived and analyzed in [1]. Values computed from Formulas (1.7) coincide for $\psi = \psi_0$ with those of the spherically symmetric solution along characteristic AB, while for ψ close to ψ_0 ($\psi - \psi_0 < 0$) a compression wave is generated which for $\gamma < \gamma_\varkappa$ becomes a shock wave. For $\psi \to 0$ (1.7) matches with the Prandtl-Meyer solution (1.6).

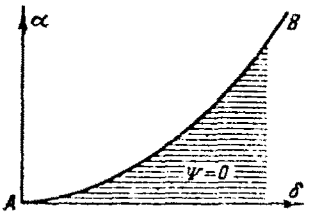

Fig. 2

With the use of asymptotics (1.7) the solution for the vicinity of point A will be analyzed in Section 2 also for finite values of δ ($0 < \delta < \frac{1}{2}\pi$) when $\psi \to 0$. Thus the complete neighborhood of point A (α is small) consists of three areas as follows :-

first, in which the solution coincides with the spherically symmetric solution, it is bounded by the detonation wave front

$$\alpha \approx 2(\varkappa + 1)\sqrt{h}\,\delta$$

and characteristic AB,

second, in which α is of the order of δ^3

$${}^{64}/_{81}\, \varkappa h^{1.5} > \psi = \alpha/\delta^3 > 0$$

this is the transition from a rarefaction to a compression wave,

third, the area of the rarefaction wave terminating at the free boundary which corresponds to finite values of δ ($0 < \delta < \frac{1}{2}\pi$).

The gas dynamics in the neighborhood of point A analyzed in Section 2 define the end part of the free boundary AK. From the Prandtl-Meyer solution (1.6) follows that for small values of α the free boundary coincides in the $\xi\eta$-plane with half-line $\delta = \frac{1}{2}\pi$. Let N be the intersection point of this half-line with the γ-axis with coordinates $\delta = \frac{1}{2}\pi$, $\alpha = \alpha_k$ where

$$\alpha_k = -\frac{(\varkappa + 1)\sin\gamma}{\cos(\gamma + {}^1/_2 \sqrt{h}\,\pi)} \tag{1.8}$$

Line AK obviously coincides in the $\xi\eta$-plane with the free boundary rectilinear part. Point K, depending on angle γ and exponent $\varkappa$, either coincides with N, or lies within segment AN.

It will be shown in Section 3 that when $\varkappa \geq 2$ and $\gamma \geq \gamma_k(\varkappa)$ (conversely $\varkappa \leq 2$ and $\gamma \lesssim < \gamma_k(\varkappa)$), then point K coincides with point N. Otherwise point K lies strictly within segment AN (Fig. 1, *a* and *b*). Angle $\gamma_k(\varkappa)$ is defined by Eq.

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*) It is assumed that $h < 9$ (i.e. $\varkappa > 1.25$) with which $\frac{1}{2}\sqrt{h}\,\pi < 1.5\pi$, and the rarefaction wave does not overtake the detonation wave front.

$$\alpha_k = \varkappa \sqrt{h} \tag{1.9}$$

Note. If the characteristics passing through any arbitrarily small neighborhood of point P do not intersect a sufficiently small neighborhood of point A, then point P does not belong to AK. Point N is reached by the flow line which is the η-axis. Hence, point N cannot belong to AK, and the coincidence of AN and AK is to be interpreted as a coincidence of the half-intervals AN and AK. Position of the free boundary and the gas dynamics in its vicinity, but outside AK can only be determined by a numerical computation of partial differential equations.

2. Asymptotics of gas-dynamic functions in the neighborhood of point A. At this point $\alpha = 0$, and the magnitude of angle δ varies from zero up to a certain value δ_0 for which f vanishes. Hence, when analyzing the asymptotic behavior in the vicinity of point A it is to be assumed that α is small and angle δ is finite. Immediately outside domain G (within domain G the solution is spherically symmetric) there exists in the vicinity of point A an area where α is of the order of δ^3 for which the solution is of the form (1.7):

$$v_\alpha = v_{\alpha_0}(\psi)\,\delta + v_{\alpha_1}(\psi)\,\delta^3 + \cdots, \quad v_\delta = v_{\delta_0}(\psi) + v_{\delta_1}(\psi)\,\delta^2 + \cdots$$
$$f = f_0(\psi) + f_1(\psi)\,\delta^2 + \cdots, \quad \psi = \alpha/\delta^3 \qquad (\alpha \to 0, \quad \delta \to 0)$$

Equations defining functions of ψ were derived and analyzed in [1], where it was shown that for small δ and $\psi \to 0$ the value of the gas-dynamic functions are defined by Formulas

$$v_\alpha \approx \varkappa \sqrt{h}\,\delta + \left[-{}^1\!/_6\varkappa \sqrt{h} + C\psi^{6/7} - \psi/(\varkappa+1)\right]\delta^3, \ \ldots$$
$$v_\delta \approx \varkappa + \left(-{}^1\!/_2\varkappa + \frac{3C\psi^{6/7}}{13\sqrt{h}}\right)\delta^2, \qquad f \approx \varkappa^2 - \left(\varkappa^2 + \frac{3\varkappa(\varkappa-1)\,C\psi^{6/7}}{13\sqrt{h}}\right)\delta^2 \tag{2.1}$$

Here C is a constant of integration. A transition to finite values of δ is implied when $\psi \to 0$. Hence, the asymptotics of the gas-dynamic functions in the area of finite values are to be sought in the form

$$v_\alpha = v_{\alpha_0}(\delta) + \alpha^{6/7} v_{\alpha_1}(\delta) + \alpha v_{\alpha_2}(\delta) + \cdots, \qquad v_\delta = v_{\delta_0}(\delta) + \alpha^{6/7} v_{\delta_1}(\delta) + \alpha v_{\delta_2}(\delta) + \cdots \tag{2.2}$$
$$f = f_0(\delta) + \alpha^{6/7} f_1(\delta) + \alpha f_2(\delta) + \cdots$$

The same method was used throughout this paper for deriving the systems of differential equations for the determination of functions with identical subscripts. It consisted of a substitution of asymptotic formulas into (1.5) and equating to zero of coefficients of equal powers of the small parameter. For $\delta = 0$ the values of functions are defined by asymptotics (2.1). The system and the initial data defining functions with subscript 0 are

$$v_{\delta_0}\left(v_{\alpha_0}'/\sqrt{h} - v_{\delta_0}\right) = 0, \qquad v_{\delta_0}\left(v_{\alpha_0}'/\sqrt{h} + v_{\alpha_0}\right) + f_0'/(\varkappa-1)\sqrt{h} = 0$$
$$v_{\delta_0} f_0'/\sqrt{h} + (\varkappa-1)\,f_0\left[v_{\alpha_0} + v_{\delta_0}'/\sqrt{h}\right] = 0$$
$$v_{\alpha_0} = 0, \qquad v_{\delta_0} = \varkappa, \qquad f = \varkappa^2 \qquad \text{for } \delta = 0 \tag{2.3}$$

The solution of system (2.3) represents a rarefaction wave (Prandtl-Meyer)

$$v_{\delta_0} = \varkappa\sqrt{h}\sin\delta, \qquad v_{\alpha_0} = \varkappa\cos\delta, \qquad f = \varkappa^2\cos^2\delta \tag{2.4}$$

The value of $\delta = \frac{1}{2}\pi$ corresponds to the free boundary. Equations defining functions with subscript 1 are

$$7\varkappa\cos\delta\left(v_{\alpha_1} - \sqrt{h}\,v_{\delta_1}\right) + 6\varkappa h\sin\delta\, v_{\alpha_1} + 3(h-1)\sqrt{h}\,f_1 = 0$$
$$7\varkappa\cos\delta\left(v_{\delta_1}' + \sqrt{h}\,v_{\alpha_1}\right) + \varkappa(13h-7)\sin\delta\, v_{\delta_1} + 3.5 f_1' = 0 \tag{2.5}$$
$$\varkappa(\varkappa-1)\cos^2\delta\left(7v_{\delta_1}' + 13\sqrt{h}\,v_{\alpha_1}\right) + 7\cos\delta\, f_1' - 14\varkappa\sin\delta\cos\delta\, v_{\delta_1} + (6h+12)\sin\delta\, f_1 = 0$$
$$v_{\alpha_1} \approx C\delta^{3/7}, \quad v_{\delta_1} \approx 3C\delta^{-4/7}/13\sqrt{h}, \quad f_1 \approx -3\varkappa(\varkappa-1)\,C\delta^{-4/7}/13\sqrt{h} \qquad \text{for } \delta \to 0$$

All functions with subscript 1 vanish when $\delta = \frac{1}{2}\pi$, and the following asymptotics hold

$$v_{\alpha_1} \approx -\frac{13\sqrt{h}}{7+3h} C_1 \beta^{1+3h/7}, \qquad v_{\delta_1} \approx C_1 \beta^{3h/7}, \qquad f_1 \approx \frac{\varkappa(\varkappa-1)(10h-7)}{3h+7} C_1 \beta^{1+3h/7} \tag{2.6}$$

Here $\beta = \frac{1}{2}\pi - \delta$, and C_1 is a constant of integration.

Equations defining functions with subscript 2 are

$$2\varkappa \sqrt{h} \cos\delta\, v_{\alpha_2}' - 2\varkappa \cos\delta\, v_{\delta_2} + 2\varkappa \sqrt{h} \sin\delta\, v_{\alpha_2} + (h-1) f_2 = 0$$

$$\varkappa(\varkappa-1)\cos\delta\, v_{\delta_2}' + f_2' + \varkappa(\varkappa+3)\sin\delta\, v_{\delta_2} + \varkappa(\varkappa-1)\sqrt{h}\cos\delta\, v_{\alpha_2} = 0$$

$$v_{\alpha_2} - \frac{(\varkappa+5)}{(\varkappa-1)\sqrt{h}} \operatorname{tg}\delta\, v_{\delta_2} + \frac{(h-1)(h+2)}{2\varkappa\sqrt{h}} \frac{\operatorname{tg}\delta}{\cos\delta} f_2 + 1 +$$
$$+ \frac{\varkappa\sqrt{h}\sin\delta\cos(\gamma+\sqrt{h}\delta) - \varkappa\cos\delta\sin(\gamma+\sqrt{h}\delta)}{(\varkappa+1)\sin\gamma} \tag{2.7}$$

$$v_{\alpha_2} = -1/(\varkappa+1), \qquad v_{\delta_2} = 0, \qquad f_2 = 0 \quad \text{for } \delta = 0$$

Asymptotics of these functions for $\delta \to \frac{1}{2}\pi$ are of the form

$$v_{\alpha_2} \approx \frac{C_2}{h-2}\beta^2, \qquad v_{\delta_2} \approx -\frac{C_2}{\sqrt{h}(h-2)}\beta, \qquad f_2 \approx -\frac{2\varkappa C_2}{\sqrt{h}(h-2)}\beta^2 \tag{2.8}$$

$$C_2 = 1 + \frac{\varkappa\sqrt{h}\cos(\gamma + \frac{1}{2}\pi\sqrt{h})}{(\varkappa+1)\sin\gamma}$$

Thus, for $\beta \to 0$ and small α the gas-dynamical parameters are expressed by Formulas

$$v_\alpha \approx \varkappa\sqrt{h}\cos\beta - \frac{13\sqrt{h}}{7+3h} C_1 \beta^{1+3h/7}\alpha^{6/7} + \frac{C_2}{h-2}\beta^2 + \cdots$$
$$v_\delta \approx \varkappa\sin\beta + C_1\beta^{3h/7}\alpha^{6/7} - \frac{C_3}{\sqrt{h}(h-2)}\beta\alpha + \cdots \tag{2.9}$$
$$f \approx \varkappa^2\sin^2\beta + \frac{\varkappa(\varkappa-1)(10h-7)}{3h+7} C_1\beta^{1+3h/7}\alpha^{6/7} - \frac{2\varkappa}{\sqrt{h}(h-2)} C_2\beta^2\alpha$$

Asymptotic formulas (2.9) were derived by series expansion of gas-dynamic functions in powers of α, and are therefore valid for that area of small α and β where the ratio of a term to the immediately preceding one tends to zero when $\alpha \to 0$. From the ratio of the third to the second terms of any of Formulas (2.9) we conclude that the latter hold for small values of magnitude ζ:

$$\zeta = \alpha\beta^{7-3h} \tag{2.10}$$

We note that $\zeta = \infty$ corresponds to the free boundary $\beta = 0$ $(\delta = \frac{1}{2}\pi)$ (*).

Formulas (2.9) are however valid throughout the neighborhood of point $\beta = 0$, $\alpha = 0$, i.e. for any value of ζ. This is readily derived from the following expression of functions in this neighborhood

$$v_\alpha \approx \varkappa\sqrt{h}\cos\beta + \beta^{3h-5}V_{\alpha_1}(\zeta) + \cdots, \quad v_\delta \approx \varkappa\sin\beta + \beta^{3h-6}V_{\delta_1}(\zeta) + \cdots,$$
$$f \approx \varkappa^2\sin^2\beta + \beta^{3h-5}F_1(\zeta) + \cdots \tag{2.11}$$

(Presentation (2.11) is implied by Formulas (2.9) and (2.10)).

Equations defining functions $V_{\alpha_1}(\zeta)$, $V_{\delta_1}(\zeta)$, $F_1(\zeta)$ are integrable as elementary functions. Formulas (2.11) coincide exactly with Formulas (2.9) after a substitution into the former of expressions of functions $V_{\alpha_1}(\zeta)$, $V_{\delta_1}(\zeta)$, $F_1(\zeta)$.

3. Asymptotics of gas-dynamic functions in the free boundary

*) It may be assumed that in the free boundary neighborhood $\varkappa < 2.5$ $(h > 7/3)$. In the following we shall consider values comprised in the interval $1.5 < \varkappa < 2.5$.

neighborhood. It follows from the asymptotic formulas derived in Section 2 that for small α the free boundary coincides in the $\xi\eta$-plane with half-line $\beta = 0$ ($\delta = \frac{1}{4}\pi$). In order to determine that part of segment AN (N is the intersection point of half-line $\beta = 0$ with the η-axis) which coincides with the free boundary it is necessary to analyze the behavior of gas-dynamic functions for $\beta \to 0$ and finite α, i.e. to find the expansion of these functions in powers of β

$$v_\alpha = \varkappa\sqrt{h} + \beta^2 v_{\alpha_0}(\alpha) + \beta^{1+3h/7} v_{\alpha_1}(\alpha) + \cdots \qquad v_\delta = \beta v_{\delta_0}(\alpha) + \beta^{3h/7} v_{\delta_1}(\alpha) + \cdots,$$
$$f = \beta^2 f_0(\alpha) + \beta^{1+3h/7} f_1(\alpha) + \cdots \tag{3.1}$$

Exponents of β and asymptotics of functions of α for $\alpha \to 0$ were derived from (2.9).

The free boundary rectilinear part coincides in the $\xi\eta$-plane with the half-interval of values of α ($0 \leq \alpha \leq \alpha_0$) in which functions of α in Formulas (3.1) are finite. It will be shown in the following that a flexure of the free boundary in the $\xi\eta$-plane occurs when any of the functions of α tend to ∞ within the interval ($0 < \alpha < \alpha_k$). The value of parameter α at the point $N - \alpha_k$ is defined by Formula (1.8).

S t a t e m e n t 3.1. When $\varkappa \geq 2$ and angle $\gamma \geq \gamma_k(\varkappa)$ (angle $\gamma_k(\varkappa)$ is defined by Eq. (1.9)), or $\varkappa \leq 2$ and angle $\gamma \leq \gamma_k(\varkappa)$, then functions of α expressed by (3.1) are finite throughout the interval ($0 < \alpha < \alpha_k$). Otherwise there exists such a value of α_0 ($0 < \alpha_0 < \alpha_k$) for which all functions of α in (3.1) become ∞.

Proof of statement 3.1 will be given in detail for functions with subscript 0, after which a proof of this Statement for functions with subscript 1 may be obtained directly.

Equations of functions with subscript 0 are

$$2\sqrt{h}\,\alpha(\varkappa\sqrt{h} - \alpha)v_{\alpha_0}' - 4v_{\delta_0}v_{\alpha_0} - 2\sqrt{h}\,v_{\delta_0}^2 + (h-1)\sqrt{h}\,\alpha f_0' = 0$$

$$\sqrt{h}\,\alpha(\varkappa\sqrt{h} - \alpha)v_{\delta_0}' - v_{\delta_0}^2 + \varkappa h v_{\delta_0} - (h-1)f_0 = 0 \tag{3.2}$$

$$\alpha(\varkappa\sqrt{h} - \alpha)f_0' + (\varkappa - 1)f_0[-\sqrt{h}\,v_{\delta_0} + \varkappa\sqrt{h} + \alpha(\varkappa\sqrt{h} - \alpha_k)(\alpha - \alpha_k)] = 0$$

$$v_{\alpha_0} \approx -\frac{\varkappa\sqrt{h}}{2} + \frac{C_2\alpha}{h-2} + \cdots, \quad v_{\delta_0} \approx \varkappa - \frac{C_2\alpha}{(h-2)\sqrt{h}} + \cdots$$

$$f_0 \approx \varkappa^2 - \frac{2\varkappa C_2}{(h-2)\sqrt{h}}\alpha + \cdots, \quad \text{при} \ \alpha \to 0$$

The system consisting of the second and third of these Eqs. may be solved independently of the first, after which the solution of the first Eq. is obtained by means of simply quadrature. A qualitative picture of this solution may thus be obtained by analyzing the last two Eqs. of (3.2).

It will be readily seen from these equations that magnitudes v_{δ_0} and f_0 are simultaneously either finite, or infinite for any values of $\alpha < \alpha_k$. Singular points of this system are $\alpha = \varkappa\sqrt{h}$ and $\alpha = \alpha_k$. Point $\alpha = \varkappa\sqrt{h}$ belongs to the integration interval ($0, \alpha_k$) then and then, when angle γ is greater than angle $\gamma_k(\varkappa)$. System (3.2) has a solution $f_0(\alpha) \equiv 0$ which does not however satisfy the asymptotics when $\alpha \to 0$, hence by virtue of uniqueness when $\alpha < \varkappa\sqrt{h}$ ($\gamma > \gamma_k(\varkappa)$), or $\alpha < \alpha_k$ ($\gamma < \gamma_k(\varkappa)$), we always have $f_0(\alpha) > 0$.

For $\gamma > \gamma_k(\varkappa)$ and $\alpha < \varkappa\sqrt{h}$ functions v_{δ_0} and f_0 are bounded, and vanish at point $\alpha = \varkappa\sqrt{h}$. In fact, coefficient $C_2 > 0$ when $\gamma > \gamma_k$, hence v_{δ_0} and f_0 decrease with small α. In order to obtain an increase in the function f_0 it is necessary for the function v_{δ_0} to become greater than $\varkappa$. But at the point at which $v_{\delta_0} = \varkappa$ the value of f_0 is smaller that $\varkappa^2$ and $v_{\delta_0}' < 0$, i.e. v_{δ_0} is still smaller than $\varkappa$. It follows from the boundedness of the functions v_{δ_0} and f_0 that they vanish at the point $\alpha = \varkappa\sqrt{h}$, and

$$v_{\delta_0} \approx A_1(\varkappa\sqrt{h} - \alpha), \qquad f_0 \approx A_2(\alpha - \varkappa\sqrt{h})^{2(\varkappa-1)} \tag{3.3}$$

It is readily seen that constants of integration A_1 and A_2 must be positive. From the first of Eqs. of system (3.2) follows that the function $v_{\delta_0} \to$ const when $\alpha \to \varkappa\sqrt{h}$.

P r o o f o f S t a t e m e n t 3.1. When $\gamma = \gamma_k(\varkappa)$ coefficients C_2 vanishes, $\alpha_k = \varkappa\sqrt{h}$, and the looked for solution of system (3.2) is

$$v_{\delta_0} = \varkappa, \qquad f_0 = \varkappa^2 \tag{3.4}$$

We also write down this solution of system (3.2) for $\varkappa = 2$ (γ is arbitrary)

$$v_{\delta_0} = \frac{\alpha_k(\alpha - \varkappa\sqrt{h})}{\sqrt{h}(\alpha - \alpha_k)}, \qquad f_0 = v_{\delta_0}^2 \tag{3.5}$$

Generally it is sufficient to investigate the field of integral curves along curve (L) specified by Eqs.

$$v_{\delta_0} = \varkappa + \frac{(\varkappa - 1)(\varkappa\sqrt{h} - \alpha_k)}{\sqrt{h}(\varkappa - 3)(\alpha - \alpha_k)}\alpha, \qquad f = v_{\delta_0}^2 \quad (L) \tag{3.6}$$

Let $dv_{\delta_0}/d\alpha$ define the direction of curve (L) at the point α, and v_{δ_0}' the direction of the field of integral curves at that point, and

$$\Delta = v_{\delta_0}' - \frac{dv_{\delta_0}}{d\alpha} = \frac{2(\varkappa - 1)(\varkappa - 2)(\varkappa\sqrt{h} - \alpha_k)\alpha_k\alpha}{(\varkappa - 3)^2(\alpha - \alpha_k)^2(\varkappa\sqrt{h} - \alpha)\sqrt{h}} \tag{3.7}$$

T h e c a s e o f $\varkappa \geq 2$ and $\gamma > \gamma_k(\varkappa)$. Function v_{δ_0} decreases with increasing α, vanishes for $\alpha = \varkappa\sqrt{h}$, and becomes negative when $\alpha > \varkappa\sqrt{h}$. The vanishing of the function v_{δ_0} for $\alpha = \varkappa\sqrt{h}$ would be accompanied by a negative derivative of v_{δ_0} which is not possible. Along curve L the function v_{δ_0} decreases monotonously, is negative for $\alpha = \varkappa\sqrt{h}$,

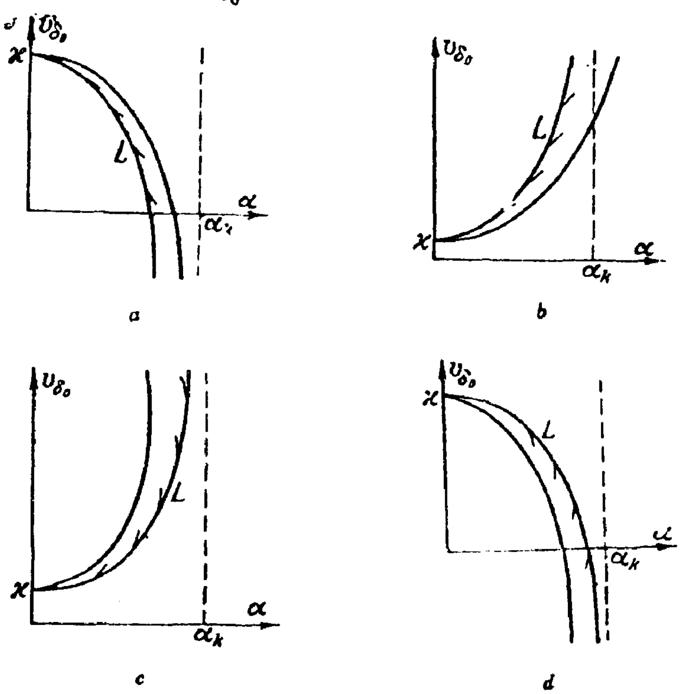

Fig. 3, *a, b, c, d*

and becomes ∞ when $\alpha = \alpha_k$. As the difference Δ is positive for $\alpha > \varkappa\sqrt{h}$, therefore v_{δ_0}, and consequently also f_0 do not become infinitely great for any values of $\alpha < \alpha_k$ (Fig. 3 *a*)(*). It will be seen from equations that for $\alpha \to \alpha_k$, the following assumptions are valid

(*Footnote carried over to next page*)

$$v_{\delta_0} \sim (\alpha - \alpha_k)^{2-\varkappa}, \qquad f \sim (\alpha - \alpha_k)^{1-\varkappa} \tag{3.8}$$

The case of $\varkappa \leq 2$ and $\gamma < \gamma_k(\varkappa)$. The magnitude of $\varkappa\sqrt{h}$ is greater than α_k, hence the function v_{δ_0} increases monotonously along curve (L), and becomes ∞ when $\alpha = \alpha_k$. Along the integral curve the magnitude of v_{δ_0} is greater than zero, as a v_{δ_0} equal to zero at a certain point would be accompanied by a positive derivative v_{δ_0}' at that point. As the difference Δ is negative, the looked for integral curve lies under curve (L), hence v_{δ_0} and f are finite for any $\alpha < \alpha_k$ (Fig. 3*b*). It will be readily seen that for $\alpha \to \alpha_k$

$$v_{\delta_0} \sim \text{const}, \qquad f_0 \sim (\alpha - \alpha_k)^{1-\varkappa} \tag{3.9}$$

The case of $\varkappa > 2$ and $\gamma < \gamma_k(\varkappa)$. The magnitude $\varkappa\sqrt{h}$ is greater than α_k, therefore the difference Δ is positive throughout the interval $(0, \alpha_k)$. Curve L becomes ∞ when $\alpha = \alpha_k$. As the point $(\alpha = 0,\ v_{\delta_0} = \varkappa,\ f = \varkappa)$ is common to curve L and the unknown integral curve, the latter must lie above curve L, and consequently must become ∞ for a certain value of α_0 $(0 < \alpha_0 \leq \alpha_k)$. Magnitude α_0 cannot coincide with α_k, as for $\alpha \to \alpha_k$ the rate of increase of v_{δ_0} along the integral curve cannot exceed $(2 - h)(\varkappa\sqrt{h} - \alpha_k)\,\alpha_k h^{-1/2}(\alpha - \alpha_k)^{-1}$ while at the same time the function v_{δ_0} increases along curve L as $(2 - h)^{-1}(\varkappa h^{1/2} - \alpha_k)\,\alpha_k h^{-1/2}(\alpha - \alpha_k)^{-1}$ when $\alpha \to \alpha_k$, and the difference of coefficients for $(\alpha - \alpha_k)^{-1}$ is negative (Fig. 3*c*).

The case of $\varkappa < 2$ and $\gamma > \gamma_k(\varkappa)$. The function v_{δ_0} decreases with increasing α, vanishes for $\alpha = \varkappa\sqrt{h}$, and then becomes negative. Along curve (L) the function v_{δ_0} is a monotonously decreasing functuon of α, is positive for $\alpha = \varkappa\sqrt{h}$, and becomes ∞ when $\alpha = \alpha_k$. As the difference Δ is negative for $\alpha > \varkappa\sqrt{h}$, hence the unknown integral curve lies under curve L, and must become ∞ for a certain value of α_0 $(\varkappa\sqrt{h} < \alpha_0 \leq \alpha_k)$.

The value of α_0 is $\alpha_0 < \alpha_k$ because the one only integral curve l passing through the point $\alpha = \alpha_k$, $v_{\delta_0} = -\infty$ does not satisfy initial data (3.2). In fact, the curve defined by the system of equations with initial data (3.2) passes through the point P $(\alpha = \varkappa\sqrt{h},\ v_{\delta_0} = 0,\ f_0 = 0)$, and in the neighborhood of that point satisfies Eqs. (3.3) with positive coefficients A_1 and A_2. Point P is a nodal point, and all integral curves passing through it satisfy in its vicinity Eqs. (3.3) for various values of coefficients A_1 and A_2. If curve l passes through the point P, then its corresponding coefficient A_1 is negative, as otherwise there would exist an integral curve lying between curves l and L, and by virtue of uniqueness having a finite value for $\alpha = \alpha_k$ which is not possible (Fig. 3*d*).

Thus, when $\varkappa > 2$, $\gamma < \gamma_k(\varkappa)$, and $\varkappa < 2$, $\gamma > \gamma_k(\varkappa)$ the value of α_0 for which the function v_{δ_0} becomes ∞ is smaller than α_k. Hence, the value of α_0 can only be determined by numerical integration. It follows from system (3.2) that for $\alpha \to \alpha_0$

$$v_{\delta_0} \approx \frac{2\sqrt{h}\,\alpha_0 (v_{\alpha_0} - \alpha_0)}{(\varkappa + 1)(\alpha_0 - \alpha)}, \quad f_0 \approx \frac{(\varkappa - 1)^2}{4} v_{\delta_0}^2, \quad v_{\alpha_0} \approx \frac{h\alpha_0^2 (v_{\alpha_0} - \alpha_0)}{(\varkappa + 1)(\alpha - \alpha_0)^2} \tag{3.10}$$

The proof for functions with subscript 0 is thus completed.

Functions with subscript 1 are defined by system

$$14\sqrt{h}\,\alpha(\varkappa\sqrt{h} - \alpha)\,v_{\alpha_1}' + 7\sqrt{h}\,\alpha(h-1)\,f_1' - 2v_{\delta_0}[(3h+7)\,v_{\alpha_1} + 7\sqrt{h}\,v_{\delta_1}] -$$
$$- 14 v_{\delta_1}(2v_{\alpha_0} + \sqrt{h}\,v_{\delta_0}) = 0$$

$$14\sqrt{h}\alpha(\varkappa\sqrt{h} - \alpha)\,v_{\delta_1}' - 2v_{\delta_1}[(3h+7)\,v_{\delta_0} - 7\varkappa h] - (h-1)(3h+7)\,f_1 = 0 \tag{3.11}$$

$$7\sqrt{h}\alpha(\varkappa\sqrt{h} - \alpha)\,f_1' - f_1\{(3h + 7\varkappa)\,v_{\delta_0} - (3\varkappa + 17)\,f_0 -$$

(*Footnote continued from previous page*)

*) All investigations will be carried out in the αv_δ-plane, and projections of curves onto this plane will be considered. In the following text the term 'curve' will be used to denote the projection of a curve on the αv_δ-plane.

$$-7(\varkappa-1)\sqrt{h}[\varkappa\sqrt{h}+\alpha(\varkappa\sqrt{h}-\alpha_k)/(\alpha-\alpha_k)]\}=0$$

$$v_{\alpha_1}\approx-\frac{13\sqrt{h}}{7+3h}C_1\alpha^{6/7},\qquad v_{\delta_1}\approx C_1\alpha^{6/7},\qquad f_1\approx\frac{2\varkappa(10h-7)}{(h-1)(3h+7)}C_1\alpha^{6/7}$$

The coefficients of this system depend on the functions with subscript 0. Consequently particular values of α in systems (3.2) and (3.11) coincide, and it is readily seen that functions with subscripts 0 and 1 are simultaneously either finite, or infinite. Thus for $\alpha \to \alpha_0$ (the third and fourth cases) the following is obtained

$$v_{\delta_1}\approx A(\alpha_0-\alpha)^{-(1+3h/7)},\qquad f_1\approx B(\alpha_0-\alpha)^{-(2+3h/7)},\qquad v_{\alpha_1}\approx C(\alpha_0-\alpha)^{-(2+3h/7)} \qquad (3.12)$$

Here A is the constant of integration of system (3.11).

$$\frac{B}{A}=-\frac{2\sqrt{h}\alpha(\varkappa\sqrt{h}-\alpha)}{h(h-1)},\qquad \frac{C}{A}=\frac{4\sqrt{h}\alpha(5h+7)}{(\varkappa+1)(10h+7)}$$

Thus the validity of Statement 3.1 is also proved for functions with subscript 1.

S t a t e m e n t 3.2. If $\varkappa \geq 2$ and $\gamma \geq \gamma_k(\varkappa)$ (conversely $\varkappa \leq 2$, $\gamma \leq \gamma_k(\varkappa)$), then the free boundary coincides with half-interval AN of half-line $\beta = 0$, where N is the intersection point of this half-line with the η-axis (Fig. 1a).

In fact, it follows from Statement 3.1 that functions of α with subscripts 0 and 1 are finite in the half-interval $(0, \alpha_k)$, and consequently f is zero in the half-interval AN.

S t a t e m e n t 3.3. If $\varkappa > 2$ and $\gamma < \gamma_k(\varkappa)$ (conversely $\varkappa < 2$ and $\gamma > \gamma_k(\varkappa)$), then the free boundary coincides with the half-interval AK of half-line $\beta = 0$, of which point K is lying strictly within interval AN (Fig. 1b). The free boundary becomes flexured at point K ($\beta = 0$, $\alpha = \alpha_0 < \alpha_k$), and

$$\beta\approx g_0^*(\alpha-\alpha_0)^{1/k}\qquad (g_0^*<0,\quad k=(3h-7)/3h) \qquad (3.13)$$

is its equation in the neighborhood of point K when $\alpha - \alpha_0 > 0$ ($\beta \to 0$, $\alpha - \alpha_0 \to 0$), and α_0 is the value of α within the interval $(0, \alpha_k)$ for which functions of α in (3.1) become ∞.

Functions of α expressed in form (3.1) are finite in the interval $(0, \alpha_0)$, hence the half-interval AK belongs to the free boundary. Point K is a singular point, and the determination of the free boundary immediately beyond that point requires a complete analysis of the gas dynamics throughout its neighborhood. Formulas (3.10) and (3.12) derived in the proof of Statement 3.1 will serve as the starting point. Substituting these into (3.1) we obtain

$$(\alpha-\alpha_0)v_\delta\approx\beta[-(h-1)h^{-1/2}\alpha_0(\varkappa\sqrt{h}-\alpha_0)+Ag^{3h/7}]+\cdots$$

$$(\alpha-\alpha_0)^2f\approx\beta^2[\alpha_0^2h^{-1}(\varkappa\sqrt{h}-\alpha_0)^2+Bg^{3h/7}]+\cdots \qquad (3.14)$$

$$(\alpha-\alpha_0)^2(v_{\alpha_0}-\varkappa\sqrt{h})\approx\beta^2[-\alpha^2(\varkappa-1)^{-1}(\varkappa\sqrt{h}-\alpha_0)+Cg^{3h/7}]+\cdots$$

$$[k=(3h-7)/3h,\ g=\beta^k/(\alpha_0-\alpha)]\qquad \text{for }\alpha_0-\alpha\to0,\quad\beta\to0$$

Formulas (3.1) are series expansions in powers of β. Formulas (3.10) and (3.12) were derived for $\alpha < \alpha_0$. Hence Formulas (3.14) hold only for small negative values of fraction $g = \beta^k/(\alpha_0 - \alpha)$. Throughout the neighborhood of point K parameter g runs over the full length of the real axis, hence functions defining the gas dynamics in this neighborhood are expressed by

$$(\alpha-\alpha_0)v_\delta\approx\beta V_\delta(g),\qquad (\alpha-\alpha_0)^2(v_\alpha-\varkappa\sqrt{h})\approx\beta^2V_\alpha(g)$$

$$(\alpha-\alpha_0)^2f\approx\beta^2F(g) \qquad (3.15)$$

The system which defines $V_\delta(g)$, $V_\alpha(g)$ and $F(g)$ is

$$\frac{dF}{dV_\delta}=\frac{\{\chi[(\varkappa-3)k-(\varkappa+1)]V_\delta+2k(F-V_\delta^2)-2\chi^2\}F}{[(h-1)(k-1)\chi+kV_\delta]F-V_\delta[\chi+kV_\delta][\chi+V_\delta]}$$

$$\frac{dV_\delta}{dg}g = \frac{-[2\chi + (\varkappa+1)V_\delta]F}{[\chi + kV_\delta]\,dF/dV_\delta + (\varkappa-1)kF} \tag{3.16}$$

$$\frac{dV_\alpha}{dg}g = -\frac{[(dF/dV_\delta)(dV_\delta/dg)g + 2F]\sqrt{h}\alpha_0 + 2(\varkappa-1)(\chi+V_\delta)V_\alpha}{(\chi+kV_\delta)(\varkappa-1)}$$

$$(\chi = \sqrt{h}\alpha_0(\varkappa\sqrt{h}-\alpha_0))$$

The initial asymptotics for $g \to 0$ are

$$V_\delta \approx -2\sqrt{h}\alpha_0(\varkappa+1)^{-1}(\varkappa\sqrt{h}-\alpha_0) + Ag^{3h/7}+\dots \tag{3.17}$$

$$F\approx(\varkappa\sqrt{h}-\alpha_0)^2\alpha_0^2h^{-1} + Bg^{3h/7}+\cdots,\quad V_\alpha \approx -(\varkappa-1)^{-1}(\varkappa\sqrt{h}-\alpha_0)^2\alpha_0^2 + Cg^{3h/7}+\dots$$

For $\dot{\alpha} < \alpha_0$ the line $g = 0$ corresponds to the free boundary along which the following two conditions must be fulfilled:-

1) pressure at the free boundary must be zero, i.e. $f = 0$,

2) the free boundary is a line of flow expressed by equation

$$\frac{d\alpha}{d\beta} = -\sqrt{h\alpha}\,\frac{v_\alpha - \alpha}{v_\delta} \tag{3.18}$$

or in terms of variables g, β by

$$\frac{dg}{d\beta} = \frac{g}{\beta}\left[k + \sqrt{h}\alpha_0\frac{\varkappa\sqrt{h}-\alpha_0+\beta^{2(1-k)}g^2V_\alpha(g)}{V_\delta(g)}\right] \tag{3.19}$$

Hence for $\alpha > \alpha_0$ the free boundary in the neighborhood of point $\beta = 0$, $\alpha = \alpha_0$ is represented either by line $g = 0$ when $Fg^{2/k} \to 0$ for $g \to 0$ $(g > 0)$, because of $f = (\alpha - \alpha\ )^{2(1-k)/k} g^{2/k}F$, or by line $g = g_0$ $(g_0 > 0)$ along which functions F and V_δ assume fixed values

$$F = 0, \qquad V_\delta = -\sqrt{h}\alpha_0(\varkappa\sqrt{h}-\alpha_0)/k \tag{3.20}$$

Integration of system (3.17) has to be obviously carried out up to that value of g which for $\alpha > \alpha_0$ corresponds to the free boundary.

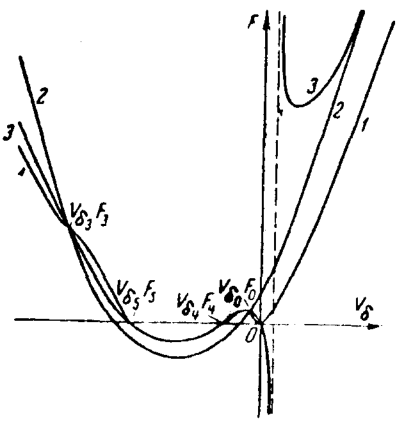

Fig. 4*a*

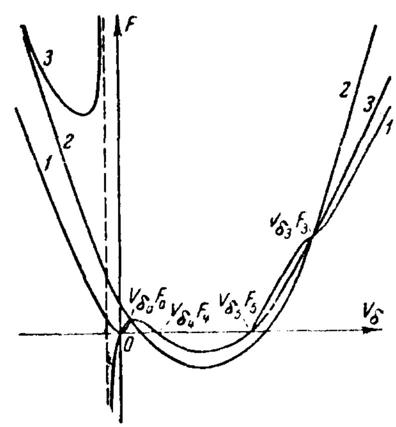

Fig. 4*b*

The first Eq. of system (3.16) may be integrated independently of the second and third Eqs., after which the integration of the latter is reduced to quadratures. Isoclines 2 and 3 of the zero and infinity of this equation respectively, and the looked for integral curve *I* are shows on Fig. 4*a* and 4*b* for the case of $\varkappa > 2$, $\gamma < \gamma_k(\varkappa)$, and $\varkappa < 2$, $\gamma > \gamma_k(\varkappa)$ respectively.

The equation has six singular points, viz.

the initial point of integration which is a saddle point

$$V_{\delta_0} = -2/\varkappa + 1\, \sqrt{h}\alpha_0 (\varkappa \sqrt{h} - \alpha_0), \qquad F_0 = h^{-1} (\varkappa \sqrt{h} - \alpha_0)^2 \alpha_0^2$$

subcritical nodal points

$$V_{\delta_1} = 0, \quad F_1 = 0; \qquad V_{\delta_2} = \pm\infty, \quad F_2 = \infty$$

nodal points

$$V_{\delta_3} = -(h-1) \sqrt{h}\alpha_0 (\varkappa \sqrt{h} - \alpha_0) [(h-2)k]^{-1}, \qquad F_3 = {}^1/_4 (\varkappa - 1)^2 V_{\delta_3}^2$$

$$V_{\delta_4} = -\sqrt{h}\alpha_0 (\varkappa \sqrt{h} - \alpha_0), \qquad F_4 = 0$$

a saddle end point

$$V_{\delta_5} = -\sqrt{h}\alpha_0 k^{-1} (\varkappa \sqrt{h} - \alpha), \qquad F_5 = 0$$

Here line $g = g_3$, V_{δ_3} the corresponding point, and F_3 a characteristic. In fact, the equation of one of the family of characteristics (a bunch of characteristics emanating from point A forms another family) is

$$\frac{d\alpha}{d\delta} = \sqrt{h}\alpha \frac{(v_\alpha - \alpha)^2 - f}{v_\delta (v_\alpha - \alpha) + \sqrt{f\,[(v_\alpha - \alpha)^2 + v_\delta^2 - f]}} \tag{3.21}$$

If the line g = const belongs to this family, then values of functions corresponding to this line must satisfy Eq.

$$-k\,[V_\delta \pm \sqrt{F}] = \sqrt{h}\,\alpha_0 (\varkappa \sqrt{h} - \alpha_0) \qquad (+\sqrt{F} \text{ for } \varkappa > 2,\ -\sqrt{F} \text{ for } \varkappa < 2) \tag{3.22}$$

A direct check will show that V_{δ_3} and F_3 satisfy (3.22).

N o t e. Statements that a line is a characteristic, or a free boundary are to be understood as meaning that equations and conditions relevant to the neighborhood of point $\alpha = \alpha_0$, $\beta = 0$ ($\alpha - \alpha_0$ and β are infinitely small magnitudes) are satisfied with respect to their predominant terms.

A particular solution of system (3.16) satisfying the asymptotics for $g \to 0$ may be written as follows

$$F = \left[(\varkappa - 1)\frac{V_\delta}{2}\right]^2, \qquad g = \frac{2a}{(\varkappa+1)V_\delta}\left[\frac{2a + (\varkappa+1)V_\delta}{A\ (\varkappa+1)}\right]^{1-k}$$

$$V_\alpha = {}^1/_2 (\varkappa^2 - 1)(k-1)\left[\frac{a + kV_\delta}{2a + (\varkappa+1)V_\delta}\right]^\nu V_\delta^2 \int \frac{[2a + (\varkappa+1)V_\delta]^{\nu-1}}{[a + kV_\delta]^{k+1}}\, dV_\delta \tag{3.23}$$

$$\nu = \frac{2(1-k)(\varkappa-1)}{(\varkappa+1) - 2k}, \qquad a = \sqrt{h}\,\alpha\,(\varkappa\sqrt{h} - \alpha)$$

Values of A obtained by numerical calculations ($A > 0$ for $\varkappa > 2$, and $A < 0$ for $\varkappa < 2$) show that the absolute value of V_δ decreases for small negative g. Hence the integral curve path is built up as follows (Fig. 4*a* and *b*): from point V_{δ_0}, F_0 corresponding to $g = 0$ ($\alpha < \alpha_0$) the curve runs towards the coordinate origin $V_{\delta_1} = 0$, $F_1 = 0$ which corresponds to $g = \infty$, after which it reaches point $V_{\delta_2} = \pm\infty$, $F_2 = \infty$ (the plus sign is for $\varkappa > 2$, and the minus for $\varkappa < 2$) corresponding to $g = 0$ ($\alpha > \alpha_0$). It will be readily seen that in this case $F \approx g^{2/k}$, hence

$$f \approx (\alpha - \alpha_0)^{2(1-k)/k}$$

therefore the line $g = 0$ does not correspond to the free boundary. Consequently, the point corresponding to the latter is

$$V_{\delta_5} = \sqrt{h}\,\alpha\,(\varkappa\sqrt{h} - \alpha)\,/\,k, \quad F_5 = 0$$

From the infinitely remote point V_{δ_2}, F_2 the curve reaches nodal point V_{δ_3}, F_3 along a separate branch. Its direction at this point is given by

$$\frac{dF}{dV_\delta}=-\frac{6h\sqrt{h}\,\alpha_0(\varkappa\sqrt{h}-\alpha_0)}{(h-2)(3h-7)(h-1)} \tag{3.24}$$

Since point V_{δ_5}, F_5 corresponding to the free boundary does not belong to the integral curve (3.23), the solution has a weak discontinuity at point V_{δ_3}, F_3, which is admissible as point V_{δ_3}, F_3 corresponds to the characteristic.

Point V_{δ_5}, F_5 is a saddle point. One of the saddle separatrices coincides with line $F = 0$.

The second separatrix emanates from point V_{δ_5}, F_5 with slope

$$\frac{dF}{dV_\delta}=\frac{\varkappa(1-k)(\varkappa\sqrt{h}-\alpha_0)}{\sqrt{h}\,\alpha_0 k\,[(h-1)\,k-h]} \tag{3.25}$$

and reaches point V_{δ_3}, F_3 tangentially to the direction of the common whisker

$$\frac{dF}{dV_\delta}=\frac{-3\alpha_0 h^{3/2}(\varkappa\sqrt{h}-\alpha_0)(13h^2-34h+28)}{(h-2)(3h-7)(h-1)(10h-14)} \tag{3.26}$$

The separatrix u corresponds to the integral curve final segment. Values of V_α along this segment may be derived by integrating the following Eq. (particular to the second and third of Eqs. of system (3.16))

$$\frac{dV_\alpha}{dV_\delta}=-\frac{1}{(a+kV_\delta)[2a+(\varkappa+1)V_\delta](\varkappa-1)}\left\{[2a+(\varkappa+1)V_\delta]\,F\frac{dF}{dV_\delta}+\right.$$

$$\left.+[2F+2(\varkappa-1)(a+V_\delta)]\left[(a+kV_\delta)\frac{dF}{dV_\delta}+(\varkappa-1)kF\right]\right\} \tag{3.27}$$

As this equation is linear, and at point V_{δ_5}, $V_\alpha(V_{\delta_5})$ its derivative is

$$\frac{dV_\alpha}{dV_\delta}=\frac{1}{[2a+(\varkappa+1)V_\delta]\,k(\varkappa-1)}\left\{-[2a+(\varkappa+1)V_\delta]\frac{dF}{dV_\delta}+\right.$$

$$\left.+2k(\varkappa-1)\varkappa(a+V_\delta)V_\delta\right\}\frac{dF}{dV_\delta} \tag{3.28}$$

hence V_α is obviously everywhere finite.

Along the looked for integral curve the corresponding angle β becomes negative beyond point $V_{\delta_2}=\infty$, $F_2=\infty$, $g=0$ $(\alpha>\alpha_0)$, therefore we substitute in this area new variables for g

$$g^*=\beta(\alpha-\alpha_0)^{-1/k} \tag{3.29}$$

By integrating the second of Eqs. of system (3.16) along the final segment from V_{δ_3}, F_3 to V_{δ_5}, F_5, we obtain the value of $g_0{}^*$ which corresponds to the free boundary. This value will obviously be negative and different from either zero, or negative infinity, i.e. the free boundary in the $\xi\eta$-plane is curvilinear for $\alpha>\alpha_0$, and at the point $\alpha=\alpha_0$, $\beta=0$ tangent to line $\beta=0$ (Fig. 1*b*).

Thus, the equation of the free boundary in the neighborhood of the point K may be expressed by Eqs. $\beta=0$ for $\alpha\le\alpha_0$, and $\beta=g_0{}^*(\alpha-\alpha_0)^{1/k}$ when $\alpha\ge\alpha_0$.

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